Resit Exam
SOLID MECHANICS (NASM)
January 30, 2020, 08:30–11:30 h

This exam comprises four problems, for which one can obtain the following points:

Question	# points
1	1 + 0.5 + 1 = 2.5
2	1.5 + 1.5 = 3
3	1 + 1.5 + 2 = 4.5
4	1.5 + 1 = 2.5

The grade is calculated as $9 * (\text{\# points})/12.5 + 1$.

Question 1 For any stress state given by the components σ_{ij}, the principal stresses σ_i can be uniquely determined from an eigenvalue problem, as described in Sec. 2.6. Let us now consider the opposite.

a. What is the level of indeterminacy of the full stress state, given the principal stresses? In other words, what can be uniquely determined and what cannot?

b. Suppose all three principal stresses are increased by the stress value Σ. How does this change the stress components in non-principal stress directions?

c. Consider a state of plane strain (in the $\boldsymbol{e}_3$-direction) for which the principal stresses in the $\boldsymbol{e}_1$-$\boldsymbol{e}_2$ plane are given. With just this information, can all components of the three-dimensional stress tensor be determined? Explain your answer.

Question 2

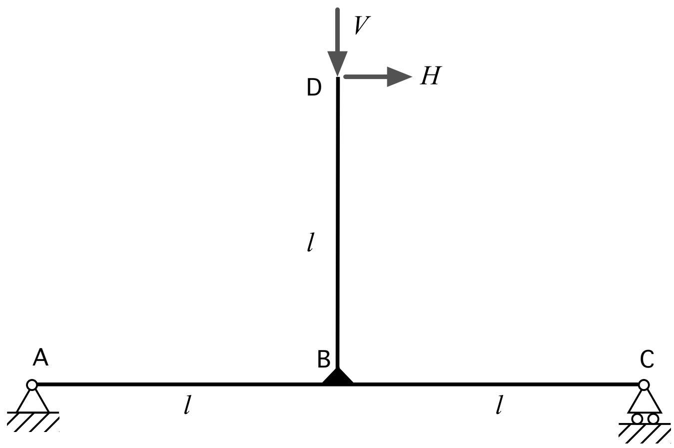
a. The elastic energy stored in an isotropically elastic body B under uniaxial tension (with strain ε) is given by $U = \frac{1}{2}\int_B E\varepsilon^2 \, \mathrm{d}V$, where E is Young's modulus. What is the elastic energy in a slender beam of length L and with moment of inertia I, when subjected to a curvature $\kappa(x)$ $(0 \leq x \leq L)$?

b. Show that the internal virtual work in such a beam can be written as

$$\delta W_{\text{in}} = \int_0^L M(x)\delta w''(x) \, \mathrm{d}x ,$$

when $w(x)$ is the deflection of the beam and $M(x)$ is the bending moment along the beam.

Question 3 A $\perp$-like spring consists of three legs having length l and bending stiffness EI. It is mounted on simple supports at the ends A and C, and is loaded at the remaining end D. A horizontal force H gives rise to a horizontal displacement u_D, a vertical force by a vertical displacement v_D, giving rise to two stiffnesses. The question is which of the two stiffnesses, H/u_D or V/v_D, is largest.



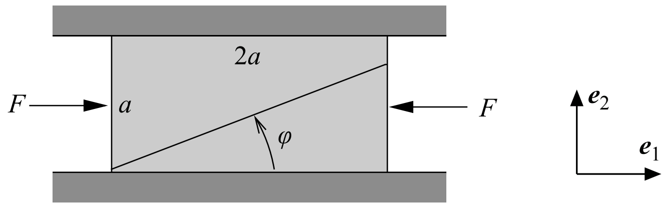
a. Compute the vertical stiffness V/v_D, neglecting axial deformation of the bar BD.

b. Determine the distribution of the bending moment in ABC caused by the horizontal force H. Specify the boundary conditions along ABC (these are needed to solve the beam equation and are helpful in using the 'forget-me-nots').

c. Compute the horizontal stiffness H/u_D.

PS. You may either solve the beam equation or use the 'forget-me-nots'. In either case, you may want to make use of (point) symmetry to make your life easier.

Question 4 Consider a block of material of dimensions $a \times 2a$ subjected to plane strain conditions perpendicular to the plane of the picture. The block is loaded by a horizontal compressive force F in the horizontal direction, which induces a uniform stress state when edge effects are neglected.



a. First consider the situation in the above figure where the block is being constrained against deformation in the vertical direction. There is no friction between the block and the two constraint platens. The material is linearly elastic and isotropic with Young's modulus E and Poisson ratio ν. Determine the three-dimensional stress state.

b. When the force becomes high enough, yield will start. Express the resolved shear stress on a plane inclined at the angle φ as shown in terms of the given quantities. What is the most favourable orientation for slip to occur?

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Question 1

a. The principal stresses determine everything, cf., e.g., Mohr's circle in Fig. 2.10. [1]

b. Adding Σ to all principal stresses is the same as adding a hydrostatic stress of that value. Hence, in any arbitrary frame of reference, the shear components remain unchanged and the normal components increase with Σ. [0.5]

c. All $\sigma_{\alpha\beta}$ components ($\alpha, \beta = 1,2$) are specified through the in-plane principal stresses, while plane strain means $\varepsilon_{i3} = 0$. In general, we therefore cannot say anything about $\sigma_{i3} = 0$. [0.5]

   However, for an isotropic material, the conditions on strain imply that $\sigma_{\alpha 3} = 0$, but σ_{33} remains unknown as long as the constitutive equations of the material are not specified. [0.5]

Question 2

a. The body B now is a beam of length L and cross-sectional area A inside of which the uniaxial stress field is given by Eq. (3.45), i.e.

$$\sigma_{11} = E\varepsilon_{11} = E\kappa x_2 \,.$$

   Substitution into $U = \frac{1}{2}\int_B \sigma_{11}\varepsilon_{11}\,\mathrm{d}V$, using the shortcut $x := x_1$ as on page 54 of the lecture notes, then yields: [0.5]

$$\begin{aligned} U &= \frac{1}{2}\int_0^L \int_A E{\varepsilon_{11}}^2 dx dx_2 dx_3 \\ &= \frac{1}{2}\int_0^L \int_A E\kappa^2 {x_2}^2 dx dx_2 dx_3 \\ &= \frac{1}{2}\int_0^L E\kappa^2 \left(\int_A {x_2}^2 dx_2 dx_3\right) dx \\ &= \frac{1}{2}\int_0^L EI\kappa^2 dx \end{aligned}$$

[1]

b. At any rate, it is important to note that $\kappa = w''(x)$ and $M(x) = EIw''(x)$. With this, there are, at least, two acceptable routes:

– One can start from the general expression (2.24), i.e. $\delta W_{\text{in}} = \int_B \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V$, and substitute the same stress and strain fields as in (a):

$$\begin{aligned}\delta W_{\text{in}} &= \int_0^L \int_A E\varepsilon_{11}\delta\varepsilon_{11} dx dx_2 dx_3 \\ &= \int_0^L E\kappa\,\delta\kappa \left(\int_A {x_2}^2 dx_2 dx_3\right) dx \\ &= \int_0^L EI\kappa\,\delta\kappa dx \\ &= \int_0^L M\delta w'' dx\,.\end{aligned}$$

– Alternatively, we could observe that the answer of (a) can be rewritten as

$$U = \frac{1}{2}\int_0^L M\kappa dx$$

and to draw on the analogy between $U = \frac{1}{2}\int_B \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V$ and $\delta W_{\text{in}} = \int_B \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V$ to conclude that for a bent beam

$$\delta W_{\text{in}} = \int_0^L M\delta\kappa dx = \int_0^L M\delta w'' dx\,.$$

1.5

Question 3

a. Since the axial deformation can be neglected, this subproblem has the same answer as when V is applied directly at point B. That is, this is essentially a three-point bending problem. According to the solution of Exercise 3.22 in the lecture notes, this problem can be reduced to a cantilever of length l that is clamped at one end (say, point B) and loaded at the opposite end by a point load $V/2$. According to the second forget-me-not (see Fig. 3.6), the deflection at B, and hence at D, is given by

$$v_D = \frac{V/2l^3}{3EI} = \frac{Vl^3}{6EI}\,.$$

Thus,

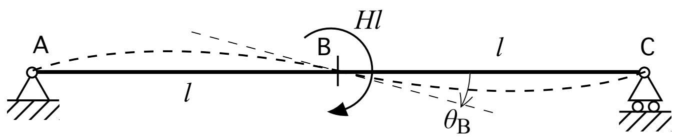
$$\frac{V}{v_D} = \frac{6EI}{l^3}$$

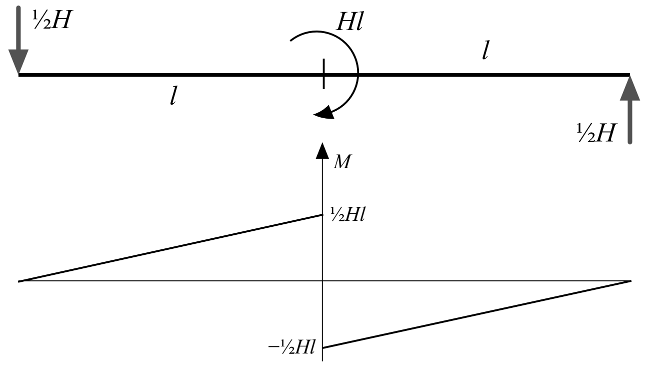
1

b. The horizontal force H at D induces a horizontal force in ABC, which is irrelevant for the deformation, plus an external torque of Hl at B:

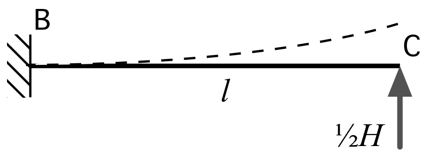
The supports at A and C give vertical reaction forces of magnitude $\frac{1}{2}H$, in opposite directions (see below), in order to maintain equilibrium. Free-body diagrams starting at either of the ends of ABC give linear distribution of the bending moment, with a jump of Hl at the center due to the external torque. 0.5

1





c. As hinted in the PS of the question, the problem under (b) is centrosymmetric around point B: that is, rotation about 180° leaves the problem unchanged, which implies that B does not move; there is only a non-zero rotation θ_B about B. Because of this, we only need to be analyze half of the problem, for instance, the part BC. The rotation θ_B is as yet unknown. If, for the moment, we clamp BC at B, point C will deflect upward by an amount $w_C = (H/2)l^3/(3EI) = Hl^3/(6EI)$, by virtue of the second forget-me-not, as illustrated in the following figure: 0.5



However, in the actual problem, see sketch under (b), C does not displace in the vertical direction. This can be eliminated by adding a rigid-body to rotation to the half-problem sketched above equal to w_C/l in the clockwise direction. This same rotation applies to the beam at B, thus yielding

$$\theta_B = \frac{Hl^2}{6EI}.$$

0.5

Knowing the rotation at this end of BD, the total displacement of D can be written as

$$u_D = \theta_B l + u_{BD},$$

where $u_{BD} = Hl^3/(3EI)$ is the deflection of BD due its bending (again, second forget-me-not). 0.5

Altogether,

$$u_D = \frac{Hl^3}{6EI} + \frac{Hl^3}{3EI} = \frac{Hl^3}{2EI},$$

so that the horizontal stiffness,

$$\frac{H}{u_D} = \frac{2EI}{l^3}\,,$$

is exactly three times the vertical stiffness V/v_D. 0.5

Question 4

a. $\sigma_{11} = -F/a$, $\sigma_{12} = 0$, $\sigma_{22} = -\frac{\nu}{1-\nu}F/a$, $\sigma_{33} = \nu\sigma_{\kappa\kappa} = -\frac{\nu}{1-\nu}F/a$, $\sigma_{\alpha 3} = 0$. 1.5

b. The resolved shear stress is given by

$$\tau = \frac{1}{2}(\sigma_{22} - \sigma_{11})\sin 2\varphi = \frac{1-2\nu}{2-2\nu}\frac{F}{a}\sin 2\varphi\,.$$

and has its maximum value at $\varphi = 45°$. 1